

# Sobolev Freud polynomials

Mohamed BOUALI

## Abstract

We investigate the uniform asymptotic of some Sobolev orthogonal polynomials. Three term recurrence relation is given, moreover we give a recurrence relation between the so-called Sobolev orthogonal polynomials and Freud orthogonal polynomials.

## 1 Introduction

During the past few years, orthogonal polynomials with respect to an inner product involving derivatives (so-called Sobolev orthogonal polynomials) have been the object of increasing number of works (see, for instance [1], [5], [6], [4], [7], [8]). Recurrence relations, asymptotics, algebraic, differentiation properties and zeros for various families of polynomials have been studied. In this paper we study a connection between a particular case of non-standard orthogonal polynomials.

For  $\lambda_1, \lambda_2 \geq 0$ , we defined the inner product

$$\langle f, g \rangle_S = \int_{\mathbb{R}} f(x)g(x)e^{-x^4} dx + \lambda_1 f(0)g(0) + \lambda_2 f'(0)g'(0),$$

We denote also by  $\|\cdot\|_S$  the norm associate to the inner product  $\langle \cdot, \cdot \rangle_S$ . Let  $Q_n$  be the sequence of orthogonal polynomial with respect to  $\langle \cdot, \cdot \rangle_S$ . We denoted  $\widehat{k}_n = \|Q_n\|_S^2 = \langle Q_n, Q_n \rangle_S$ .

Let  $P_n$  be the sequence of monic polynomials orthogonal with respect to the inner product  $\langle f, g \rangle_F = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^4} dx$  : They have been considered by Nevai [14,15]. These polynomials satisfy a three-term recurrence

relation

$$xP_n(x) = P_{n+1}(x) + c_n P_{n-1}(x),$$

with initial conditions  $P_0(x) = 1$  and  $P_1(x) = x$ ; where the parameters  $c_n$  satisfy a non-linear recurrence relation (see [4])

$$n = 4c_n(c_{n+1} + c_n + c_{n-1}), \quad n \geq 1,$$

with  $c_0 = 0$  and  $c_1 = \Gamma(3/4)/\Gamma(1/4)$ . Moreover the polynomial  $P_n$  satisfies the recurrence relation

$$P'_n(x) = nP_{n-1}(x) + d_n P_{n-3}(x), \quad n \geq 3$$

where  $d_n = 4k_n/k_{n-3}$ , with  $k_n = \|P_n\|_F^2 = \int_{-\infty}^{+\infty} (P_n(x))^2 e^{-x^4} dx$ . One can see from the three-term recurrence relation that

$$k_n = c_n k_{n-1}, \tag{1.1}$$

and

$$d_n = 4c_n c_{n-1} c_{n-2}.$$

### Lemma 1.1

1. For all polynomials  $P, Q$ ,  $\langle x^m P, Q \rangle_S = \langle P, x^m Q \rangle_S = \langle x^m P, Q \rangle_F = \langle P, x^m Q \rangle_F$ , where  $m \geq 1$  if  $\lambda_2 = 0$ , and  $m \geq 2$  if  $\lambda_2 > 0$ .
2. For a polynomial  $Q$ , we denote  $\widetilde{Q}(x) = Q(-x)$ . For all polynomials  $P, Q$ , we have  $\langle \widetilde{Q}, P \rangle_S = \langle Q, \widetilde{P} \rangle_S$ .
3.  $Q_n(-x) = (-1)^n Q_n(x)$ .

#### Proof.

The proof of the first item is easy.

2) Using the symmetry of the Freud inner product  $\langle \widetilde{Q}, P \rangle_F = \langle Q, \widetilde{P} \rangle_F$  and the fact that  $\widetilde{P}'(0) = -P'(0)$

$$\begin{aligned} \langle \widetilde{Q}, P \rangle_S &= \langle \widetilde{Q}, P \rangle_F + \lambda_1 Q(0)P(0) - \lambda_2 Q'(0)P'(0) \\ &= \langle Q, \widetilde{P} \rangle_F + \lambda_1 Q(0)P(0) - \lambda_2 Q'(0)P'(0) \\ &= \langle Q, \widetilde{P} \rangle_F + \lambda_1 Q(0)\widetilde{P}(0) + \lambda_2 Q'(0)\widetilde{P}'(0) \\ &= \langle Q, \widetilde{P} \rangle_S \end{aligned}$$

3) From the first step we have by orthogonality  $\langle \widetilde{Q}_n, P \rangle_S = 0$ , for all polynomials  $P$  with  $\deg(P) \leq n-1$ .

Hence  $\widetilde{Q}_n(x) = \alpha_n Q_n(x)$ , equaling the leading coefficient we obtain  $\alpha_n = (-1)^n$ .

## 2 Case $\lambda_2 = 0$

**Proposition 2.1** *The polynomials  $P_n$  and  $Q_n$  are related by*

$$xP_n(x) = Q_{n+1}(x) + a_n Q_{n-1}(x), \quad n \geq 1$$

$$xQ_n(x) = P_{n+1}(x) + b_n P_{n-1}(x), \quad n \geq 1$$

$$Q_0(x) = 1, \quad Q_1(x) = x, \text{ where } a_n = \frac{k_n}{\widehat{k}_{n-1}}, \quad b_n = \frac{\widehat{k}_n}{k_{n-1}}.$$

**Proof.** Since, let write

$$xP_n(x) = Q_{n+1}(x) + \sum_{k=0}^n \alpha_k Q_k(x), \quad (2.2)$$

By orthogonality one gets

$$\|Q_k\|_S^2 \alpha_k = \int_{-\infty}^{+\infty} P_n(x) Q_k(x) x e^{-x^4} dx,$$

Since for  $k \leq n-2$ , by orthogonality the integral vanishes, moreover using the symmetry of the inner product, one as

$$P_n(-x) = (-1)^n P_n(x), \quad Q_k(-x) = (-1)^k Q_k(x),$$

hence  $\alpha_n = 0$ , and  $a_n = \alpha_{n-1}$ .

$$\widehat{k}_{n-1} \alpha_{n-1} = \int_{-\infty}^{+\infty} P_n(x) x Q_{n-1}(x) e^{-x^4} dx = k_n,$$

where we used  $xQ_{n-1}(x) = P_n(x) + \dots$ . The second statement can be proved by a same argument.

**Proposition 2.2**

1.  $\frac{c_{n+2}c_{n+1}}{a_{n+2}} + a_n = c_{n+1} + c_n.$
2.  $a_{n+1}b_n = c_{n+1}c_n$
3.  $\lim_{n \rightarrow \infty} \frac{a_n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{b_n}{\sqrt{n}} = \frac{1}{2\sqrt{3}}.$

**Proof.**

1) Since  $xP_n(x) = Q_{n+1} + a_n Q_{n-1}(x)$ , hence

$$\|xP_n\|_S^2 = \|Q_{n+1}\|_S^2 + a_n^2 \|Q_{n-1}\|_S^2,$$

moreover

$$\|Q_{n-1}\|_S^2 = \frac{k_n}{a_n},$$

and  $\|xP_n\|_S^2 = \|xP_n\|_F^2$ , from the tree-term recurrence relation one gets

$$\|xP_n\|_F^2 = \|P_{n+1}\|_F^2 + c_n^2 \|P_{n-1}\|_F^2 = (c_{n+1} + c_n)k_n,$$

thus

$$\frac{k_{n+2}}{a_{n+2}} + a_n k_n = (c_{n+1} + c_n)k_n,$$

hence

$$\frac{c_{n+2}c_{n+1}}{a_{n+2}} + a_n = c_{n+1} + c_n.$$

2) We saw that  $a_{n+1} = \frac{k_{n+1}}{\widehat{k_n}}$ ,  $b_n = \frac{\widehat{k_n}}{k_{n-1}}$ , and  $k_n = c_n k_{n-1}$ , hence

$$a_{n+1}b_n = \frac{k_{n+1}}{k_{n-1}} = c_{n+1}c_n.$$

3) Let  $\lambda_n = a_n \sqrt{n}$ ,  $\sigma_n = \frac{c_{n+2}c_{n+1}}{n+1}$ ,  $\delta_n = \frac{c_{n+1}+c_n}{\sqrt{n+1}}$ . Then we obtain,

$$\frac{\sigma_n}{\lambda_{n+1}} + \frac{\sqrt{n-1}}{\sqrt{n+1}} \lambda_{n-1} = \delta_n, \quad (2.3)$$

Using the fact that  $\lim_{n \rightarrow \infty} \frac{c_n}{\sqrt{n}} = \frac{1}{2\sqrt{3}}$ , (see for instance [7], [10]) one gets,

$$\lim_{n \rightarrow +\infty} \sigma_n = \frac{1}{12}, \quad \lim_{n \rightarrow +\infty} \delta_n = \frac{1}{\sqrt{3}}.$$

Moreover since  $\lambda_n \geq 0$ , and

$$\lambda_n \leq \sqrt{\frac{n+2}{n}} \delta_{n+1},$$

thus the sequence  $\lambda_n$  is bounded. Let  $\ell$  be the limit of a subsequence. It follows from equation (2.3),

$$\frac{1}{12\ell} + \ell = \frac{1}{\sqrt{3}},$$

and the unique solution is  $\ell = \frac{1}{2\sqrt{3}}$ . Hence the unique limit of a subsequence of  $\lambda_n$  is  $\ell = \frac{1}{2\sqrt{3}}$ , then the bounded sequence  $\lambda_n$  converges to  $\ell = \frac{1}{2\sqrt{3}}$ .

The same hold for  $\frac{b_n}{\sqrt{n}}$  from the relation  $a_{n+1}b_n = c_{n+1}c_n$ .

**Theorem 2.3** *The asymptotic behavior*

$$\lim_{n \rightarrow \infty} \frac{Q_n(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \sqrt[4]{12} \frac{x\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)}{1 + \varphi^2\left(\sqrt[4]{\frac{3}{4}}x\right)},$$

hold uniformly on compact subset of  $\mathbb{C} \setminus [-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ , where

$\varphi(x) = x + \sqrt{x^2 - 1}$ , with  $\sqrt{x^2 - 1} > 0$  for  $x > 1$ , i.e., the conformal mapping of  $\mathbb{C} \setminus [-1, 1]$  onto the exterior of the closed unit disk.

**Proof.** We saw that

$$xQ_n(x) = P_{n+1}(x) + b_nP_{n-1}(x).$$

It is well-known (see [16]) that from the three-term recurrence relation of non normalizing Freud polynomials

$$xS_n(x) = \alpha_{n+1}S_{n+1}(x) + \alpha_nS_{n-1}(x),$$

we can obtain asymptotic properties of the orthonormal polynomials  $S_n$ :

Indeed, as  $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\sqrt[4]{n}} = \frac{1}{\sqrt[4]{12}}$ . We deduce (see [11])

$$\lim_{n \rightarrow \infty} \frac{S_{n-1}(\sqrt[4]{n}x)}{S_n(\sqrt[4]{n}x)} = \frac{1}{\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)},$$

uniformly on compact subsets of  $\mathbb{C} \setminus [-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ . Then, for the monic Freud polynomial  $P_n$  one gets

$$\lim_{n \rightarrow \infty} \sqrt[4]{n} \frac{P_{n-1}(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{\sqrt[4]{12}}{\varphi(\sqrt[4]{\frac{3}{4}}x)},$$

uniformly on compact subsets of  $\mathbb{C} \setminus [-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ .

Since

$$\frac{Q_n(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{1}{x} \left( \frac{P_{n+1}(\sqrt[4]{n}x)}{\sqrt[4]{n}P_n(\sqrt[4]{n}x)} + \frac{b_n}{\sqrt{n}} \frac{\sqrt[4]{n}P_{n-1}(x)}{P_n(x)} \right).$$

As  $n$  goes to infinity, one gets on every compact subsets of  $\mathbb{C} \setminus [-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ ,

$$\lim_{n \rightarrow \infty} \frac{Q_n(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{1}{x} \left( \frac{\varphi(\sqrt[4]{\frac{3}{4}}x)}{\sqrt[4]{12}} + \frac{1}{\sqrt{12}} \frac{\sqrt[4]{12}}{\varphi(\sqrt[4]{\frac{3}{4}}x)} \right).$$

A simple computation gives

$$\lim_{n \rightarrow \infty} \frac{Q_n(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{1}{\sqrt[4]{12}} \frac{1 + \left( \varphi(\sqrt[4]{\frac{3}{4}}x) \right)^2}{x \varphi(\sqrt[4]{\frac{3}{4}}x)}.$$

**Theorem 2.4** *The polynomials  $Q_n$  have all their zeros real and simple. For  $n \geq 3$  the positive zeros of  $Q_n$  interlace with those of  $P_n$ .*

**Proof.** We distinguish two cases: the even and the odd one, respectively. The proofs are similar with slight differences.

**Even case:** Let  $x_{2m,k}, k = 1, \dots, m$ , be the positive zeros of  $P_{2m}$  in increasing order, that is,  $x_{2m,1} < \dots < x_{2m,m}$ . First, we need to study the sign of the integrals

$$I_{2m,k} = \int_{-\infty}^{+\infty} Q_{2m}(x) \frac{P_{2m}(x)}{x^2 - x_{2m,k}^2} e^{-x^4} dx, \quad m \geq 2, k = 1, \dots, m,$$

We have

$$I_{2m,k} = \int_{-\infty}^{+\infty} Q_{2m}(x) \prod_{j=1, j \neq k}^{m-1} (x^2 - x_{2m,j}^2) e^{-x^4} dx = \sum_{r=0}^{m-1} b_r \int_{-\infty}^{+\infty} Q_{2m}(x) x^{2r} e^{-x^4} dx = b_0(k) \lambda_1 Q_{2m}(0),$$

where  $b_0(k) = (-1)^{m-1} \prod_{j=1, j \neq k}^{m-1} x_{2m,j}^2$ , and  $Q_{2m}(0) = -a_{2m} Q_{2m-2}(0) = (-1)^m \prod_{k=2}^m a_{2k}$ ,  
 $(a_{2k} > 0)$ , moreover  $\text{sign}(b_0) = (-1)^{m-1}$ , hence

$$\text{sign}(I_{2m,k}) = -1. \quad (2.4)$$

On the other hand, using Gaussian quadrature in all the zeros of  $P_{2m}$  and taking into account the symmetry of the polynomials  $Q_{2m}$ , the Christoffel numbers (see, for example, [7, p140])

$$\mu_{2m,i} = \frac{1}{\sum_{j=0}^{2m-1} P_j^2(x_{2m,i})}, \quad i = 1, \dots, m,$$

together with the fact

$$\frac{P'_{2m}(x_{2m,k})}{2x_{2m,k}} = \prod_{j=1, j \neq k}^m (x_{2m,k}^2 - x_{2m,j}^2),$$

we get

$$I_{2m,k} = \mu_{2m,k} Q_{2m}(x_{2m,k}) \frac{P'_{2m,k}(x_{2m,k})}{2x_{2m,k}},$$

and from (2.4) we deduce

$$\text{sign}(Q_{2m}) = -\text{sign}(P'_{2m,k}).$$

Since  $P'_{2m}(x)$  has opposite sign in two consecutive zeros of  $P_{2m}(x)$ , we deduce that it also occurs for  $Q_{2m}(x)$ , and therefore  $Q_{2m}(x)$  has one zero in each interval  $(x_{2m,k}, x_{2m+1,k})$ ,  $k = 1, \dots, m-1$  (and from the symmetry it has one zero in each interval  $(-x_{2m+1,k}, -x_{2m,k})$ ,  $k = 1, \dots, m-1$ ). Thus  $Q_{2m}(x)$  has at least  $2m-2$  real and simple zeros interlacing with those of  $P_{2m}(x)$ . Finally, as  $P'_{2m}(x_{2m,2m}) > 0$  then  $Q_{2m}(x_{2m,2m}) < 0$  and since  $Q_{2m}(x)$  is monic we deduce the existence of one zero of  $Q_{2m}(x)$  in  $(x_{2m,m}, +\infty)$  and another zero in  $(-\infty, -x_{2m,m})$ , which complete the result for the even case.

**Odd case:** Let  $m \geq 2$ ,  $0 < x_{2m+1,1} < \dots < x_{2m+1,2m}$  be the positive simple zeros of  $P_{2m+1}$ , since  $P_{2m+1}(0) = Q_{2m+1}(0) = 0$ , let define the integral

$$I_{2m+1,k} = \int_{-\infty}^{+\infty} Q_{2m+1}(x) \frac{P_{2m+1,k}(x)}{x^2(x^2 - x_{2m+1,k}^2)} e^{-x^4} dx$$

hence

$$I_{2m+1,k} = \int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} \prod_{j=1, j \neq k}^m (x^2 - x_{2m,j}^2) e^{-x^4} dx,$$

$$I_{2m+1,k} = \sum_{r=0}^{m-1} b_r(k) \int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} x^{2r} e^{-x^4} dx,$$

Since for  $1 \leq r \leq m-1$ ,

$$\int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} x^{2r} e^{-x^4} dx = \langle Q_{2m+1}, x^{2r-1} \rangle_S - \lambda_1(Q_{2m+1} x^{2r-1})|_{x=0},$$

hence by orthogonality one gets for  $1 \leq r \leq m-1$

$$\int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} x^{2r} e^{-x^4} dx = 0.$$

Thus

$$I_{2m+1,k} = b_0(k) \int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} e^{-x^4} dx, \quad (2.5)$$

since

$$xP_{2m}(x) = Q_{2m+1}(x) + a_{2m}Q_{2m-1}(x),$$

hence

$$\int_{-\infty}^{+\infty} P_{2m}(x) e^{-x^4} dx = \int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} e^{-x^4} dx + a_{2m} \int_{-\infty}^{+\infty} \frac{Q_{2m-1}(x)}{x} e^{-x^4} dx$$

thus by orthogonality  $\int_{-\infty}^{+\infty} P_{2m}(x) e^{-x^4} dx = 0$ , and

$$\int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} e^{-x^4} dx = -a_{2m} \int_{-\infty}^{+\infty} \frac{Q_{2m-1}(x)}{x} e^{-x^4} dx$$

and

$$\int_{-\infty}^{+\infty} \frac{Q_{2m+1}(x)}{x} e^{-x^4} dx = 2\Gamma\left(\frac{5}{4}\right)(-1)^m \prod_{k=1}^m a_{2k}. \quad (2.6)$$

Moreover

$$b_0(k) = (-1)^{m-1} \prod_{j=1, j \neq k}^m x_{2m+1,j}^2, \quad (2.7)$$

from equations (2.5), (2.6) and (2.7) one gets

$$\text{sign}(I_{2m+1,k}) = -1.$$

The rest of the proof is as in the even case.



### 3 Case $\lambda_2 \neq 0$

**Proposition 3.1** For all  $n \geq 1$ ,

$$xP_{2n-1}(x) = Q_{2n}(x) + a_n Q_{2n-2}(x),$$

$$x^2 P_n(x) = Q_{n+2}(x) + b_n Q_n(x) + \alpha_n Q_{n-2}(x),$$

$$x^2 Q_n(x) = P_{n+2}(x) + \sigma_n P_n(x) + \delta_n P_{n-2}(x).$$

with  $Q_0(x) = 1$ ,  $Q_1(x) = x$ , where,  $a_n = \frac{k_{2n-1}}{k_{2n-2}}$ ,  $\alpha_n = \frac{k_n}{k_{n-2}}$ ,  $b_n = \frac{\langle x^2 P_n, Q_n \rangle_S}{\langle Q_n, Q_n \rangle_S}$ ,  $\delta_n = \frac{\widehat{k}_n}{k_{n-2}}$ ,  $\sigma_n = b_n \frac{\widehat{k}_n}{k_n}$ .

**Proof.** The proof is as in proposition 1.1.

**Proposition 3.2** For all  $n \geq 1$ ,

1.  $\frac{c_{2n+1}c_{2n}}{a_{n+1}} + a_n = c_{2n} + c_{2n-1}$ .
2.  $c_{n+2}c_{n+1} + c_n c_{n-1} + (c_{n+1} + c_n)^2 = \frac{c_{n+4}c_{n+3}c_{n+2}c_{n+1}}{\alpha_{n+4}} + b_n^2 \frac{c_{n+2}c_{n+1}}{\alpha_{n+2}} + \alpha_n$ .
3.  $\sigma_n = \frac{n}{4c_n} + c_{n-2} - b_{n-2}$ .
4.  $c_{n+2}c_{n+1} \frac{\sigma_{n+2}}{b_{n+2}} + b_n \sigma_n + \alpha_n = \frac{n}{2} + \frac{1}{4}$ .
5.  $\sigma_n = \frac{\delta_n b_n}{c_n c_{n-1}}$ .
6.  $\lim_{n \rightarrow \infty} \frac{a_n}{\sqrt{2n}} = \frac{1}{2\sqrt{3}}$ ,  $\lim_{n \rightarrow \infty} \frac{b_n}{\sqrt{n}} = \frac{1}{\sqrt{3}}$ ,  $\lim_{n \rightarrow \infty} \frac{\alpha_n}{n} = \frac{1}{12}$ .
7.  $\lim_{n \rightarrow \infty} \frac{\sigma_n}{\sqrt{n}} = \frac{1}{\sqrt{3}}$ ,  $\lim_{n \rightarrow \infty} \frac{\delta_n}{n} = \frac{1}{12}$ ,  $\lim_{n \rightarrow \infty} \frac{\widehat{k}_n}{k_n} = 1$ .

**Proof.**

- 1) The first relation can be proved as the case  $\lambda_2 = 0$ .
- 2) From the second relation in the previous proposition and orthogonality one gets

$$\|x^2 P_n\|_S^2 = \widehat{k}_{n+2} + b_n^2 \widehat{k}_n + \alpha_n^2 \widehat{k}_{n-2}.$$

Using the fact that  $k_n = c_n k_{n-1}$ , and  $\widehat{k}_n = \frac{k_{n+2}}{\alpha_{n+2}}$ . One gets

$$\|x^2 P_n\|_S^2 = \left( \frac{c_{n+4} c_{n+3} c_{n+2} c_{n+1}}{\alpha_{n+4}} + b_n^2 \frac{c_{n+2} c_{n+1}}{\alpha_{n+2}} + \alpha_n \right) k_n \quad (3.8)$$

Since from the three term recurrence relation  $xP_n(x) = P_{n+1}(x) + c_n P_{n-1}(x)$ , and orthogonality we have

$$\|xP_n\|_S^2 = \|xP_n\|_F^2 = k_{n+1} + c_n^2 k_{n-1},$$

and

$$\begin{aligned} \|x^2 P_n\|_S^2 &= \|x^2 P_n\|_F^2 = \|xP_{n+1} + xc_n P_{n-1}\|_F^2 \\ &= \|xP_{n+1}\|_F^2 + c_n^2 \|xP_{n-1}\|_F^2 + 2c_n \langle xP_{n+1}, xP_{n-1} \rangle_F \\ &= k_{n+2} + c_{n+1}^2 k_n + c_n^2 k_n + c_n^2 c_{n-1}^2 k_{n-2} + 2c_n k_{n+1} \\ &= (c_{n+2} c_{n+1} + c_{n+1}^2 + c_n^2 + c_n c_{n-1} + 2c_n c_{n+1}) k_n \\ &= (c_{n+2} c_{n+1} + (c_{n+1} + c_n)^2 + c_n c_{n-1}) k_n. \end{aligned} \quad (3.9)$$

From equation (3.8), (3.9) one gets the desired result.

3) By orthogonality one gets

$$\sigma_n k_n = \langle x^2 Q_n, P_n \rangle_F,$$

since by definition of the Sobolev inner product we have

$$\begin{aligned} \langle x^2 Q_n, P_n \rangle_F &= \langle x^2 Q_n, P_n \rangle_S = \langle Q_n, x^2 P_n \rangle_S \\ &= \langle x^2 P_{n-2} - b_{n-2} Q_{n-2} - \alpha_{n-2} Q_{n-4}, x^2 P_n \rangle_S, \\ &= \langle x^2 P_{n-2}, x^2 P_n \rangle_S - b_{n-2} \langle Q_{n-2}, x^2 P_n \rangle_S - \alpha_{n-2} \langle Q_{n-4}, x^2 P_n \rangle_S \end{aligned}$$

since

$$\begin{aligned} \langle x^2 P_{n-2}, x^2 P_n \rangle_S &= \int_{-\infty}^{+\infty} x^4 P_{n-2}(x) P_n(x) e^{-x^4} dx \\ &= \frac{1}{4} \int_{-\infty}^{+\infty} P_{n-2}(x) P_n(x) e^{-x^4} dx + \frac{1}{4} \int_{-\infty}^{+\infty} x P_{n-2}'(x) P_n(x) e^{-x^4} dx \\ &\quad + \frac{1}{4} \int_{-\infty}^{+\infty} x P_{n-2}(x) P_n'(x) e^{-x^4} dx, \end{aligned}$$

by orthogonality the first and the second integral vanished. Moreover

$$\begin{aligned} \int_{-\infty}^{+\infty} x P_{n-2}(x) P_n'(x) e^{-x^4} dx &= \int_{-\infty}^{+\infty} x P_{n-2}(x) (nP_{n-1}(x) + d_n P_{n-3}(x)) e^{-x^4} dx \\ &= nk_{n-1} + d_n k_{n-2}. \end{aligned}$$

$$\langle Q_{n-2}, x^2 P_n \rangle_S = \langle x^2 Q_{n-2}, P_n \rangle_F = k_n.$$

$$\langle Q_{n-4}, x^2 P_n \rangle_S = \langle x^2 Q_{n-4}, P_n \rangle_F = 0.$$

Hence

$$\sigma_n k_n = \frac{n}{4} k_{n-1} + \frac{1}{4} d_n k_{n-2} - b_{n-2} k_n,$$

using the fact that  $k_n = c_n k_{n-1}$ , and  $d_n = 4c_n c_{n-1} c_{n-2}$ . Thus

$$\sigma_n = \frac{n}{4c_n} + c_{n-2} - b_{n-2},$$

which complete the proof of the assertion.

4) From the second recurrence relation of the proposition we have

$$\langle x^2 P_n, x^2 P_n \rangle_S = \widehat{k}_{n+2} + b_n^2 \widehat{k}_n + \alpha_n^2 \widehat{k}_{n-2}.$$

Moreover  $\langle x^2 P_n, x^2 P_n \rangle_S = \langle x^2 P_n, x^2 P_n \rangle_F$ , and

$$\begin{aligned} \langle x^2 P_n, x^2 P_n \rangle_F &= \int_{-\infty}^{+\infty} x^4 P_n^2(x) e^{-x^4} dx \\ &= \frac{1}{4} \int_{-\infty}^{+\infty} P_n^2(x) e^{-x^4} dx + \frac{1}{2} \int_{-\infty}^{+\infty} x P_n'(x) P_n(x) e^{-x^4} dx \\ &= \frac{1}{4} k_n + \frac{1}{2} n k_n, \end{aligned}$$

thus,

$$\frac{\widehat{k}_{n+2}}{k_n} + b_n^2 \frac{\widehat{k}_n}{k_n} + \alpha_n^2 \frac{\widehat{k}_{n-2}}{k_n} = \frac{1}{4} + \frac{1}{2} n,$$

since  $\frac{\widehat{k}_n}{k_n} = \frac{\sigma_n}{b_n}$ ,  $k_n = \alpha_n \widehat{k}_{n-2}$ , and  $k_n = c_n k_{n-1}$ , it follows that

$$c_{n+2} c_{n+1} \frac{\sigma_{n+2}}{\alpha_{n+2}} + b_n \sigma_n + \alpha_n = \frac{n}{2} + \frac{1}{4}.$$

Which give the desired result.

5) It is easy deduced from the three relations  $\delta_n = \frac{\widehat{k}_n}{k_{n-2}}$ ,  $\sigma_n = b_n \frac{\widehat{k}_n}{k_n}$ , and  $k_n = c_n k_{n-1}$ .

6) The first limit can be proven in the same way of the case  $\lambda_2 = 0$ , in fact we saw

$$\frac{c_{2n+1} c_{2n}}{a_{n+1}} + a_n = c_{2n} + c_{2n-1},$$

using the same argument as in the case  $\lambda_2 = 0$ , one gets  $\lim_{n \rightarrow +\infty} \frac{a_n}{\sqrt{2n}} = \ell = \frac{1}{2\sqrt{3}}$ .

To obtain the second limit, one can see from the first equation that

$$0 \leq \frac{\alpha_n}{n} \leq \frac{1}{n}(c_{n+2}c_{n+1} + c_n c_{n-1} + (c_{n+1} + c_n)^2),$$

since the right hand side converge, hence  $\frac{\alpha_n}{n}$  is bounded. And

$$\left(\frac{b_n}{\sqrt{n}}\right)^2 \leq \frac{c_{n+2}c_{n+1} + c_n c_{n-1} + (c_{n+1} + c_n)^2}{c_{n+2}c_{n+1}} \frac{\alpha_{n+2}}{n},$$

moreover the sequence  $\frac{c_n}{\sqrt{n}}$  converge and  $\frac{\alpha_n}{n}$  is bounded hence the se-

quence  $\frac{b_n}{\sqrt{n}}$  is bounded. Let  $x$  be a limit of any subsequence of  $\frac{b_{n_k}}{\sqrt{n_k}}$  and

$\ell = \frac{1}{2\sqrt{3}}$  the limit of the sequence  $\frac{c_n}{\sqrt{n}}$ . Since from the third and fourth items we have

$$\sigma_n = \frac{n}{4c_n} + c_{n-2} - b_{n-2},$$

and

$$\alpha_n = \frac{n}{2} + \frac{1}{4} - c_{n+2}c_{n+1} \frac{\sigma_{n+2}}{b_{n+2}} - b_n \sigma_n.$$

Thus

$$\alpha_n = \frac{n}{2} + \frac{1}{4} - c_{n+2}c_{n+1} \frac{1}{b_{n+2}} \left( \frac{n+2}{4c_{n+2}} + c_n - b_n \right) - b_n \left( \frac{n}{4c_n} + c_{n-2} - b_{n-2} \right).$$

Substitute the expression of  $\alpha_n$  in the first item of the proposition and letting  $k$  to infinity and use the fact that  $\frac{b_{n_k}}{\sqrt{n_k}}$  converge to  $x$  and  $\frac{c_{n_k}}{\sqrt{n_k}}$  converge to  $\ell$ , one gets

$$\begin{aligned} -6\ell^2 &= \frac{\ell^4}{-\frac{1}{2} + x(\frac{1}{4\ell} + \ell - x) + \frac{1}{x}(\frac{\ell}{4} + \ell^2 - \ell x)} + \frac{x^2\ell^2}{-\frac{1}{2} + x(\frac{1}{4\ell} + \ell - x) + \frac{1}{x}(\frac{\ell}{4} + \ell^2 - \ell x)} \\ &\quad + \frac{1}{2} + x(\frac{1}{4\ell} + \ell - x) + \frac{1}{x}(\frac{\ell}{4} + \ell^2 - \ell x), \end{aligned}$$

substitute the value of  $\ell = \frac{1}{2\sqrt{3}}$  we obtain the following equation

$$\frac{(1 + 12x^2)(1 - 4\sqrt{3}x + 18x^2 - 12\sqrt{3}x^3 + 9x^4)}{3x(2\sqrt{3} - 21x + 24\sqrt{3}x^2 - 36x^3)} = \frac{\frac{1}{9}\left(x - \frac{1}{\sqrt{3}}\right)^4(1 + 12x^2)}{3x(2\sqrt{3} - 21x + 24\sqrt{3}x^2 - 36x^3)} = 0, \quad (3.10)$$

Now we prove that such equation is correctly defined. Since the roots of the polynomial  $x(2\sqrt{3} - 21x + 24\sqrt{3}x^2 - 36x^3)$  are 0,  $\ell = \frac{1}{2\sqrt{3}}$  and two complex roots, moreover  $\frac{b_{n_k}}{\sqrt{n_k}}$  is a real sequence.

**First case**  $x = \ell = \frac{1}{2\sqrt{3}}$ . Since,

$$\sigma_{n_k} = \frac{n_k}{4c_{n_k}} + c_{n_k-2} - b_{n_k-2},$$

hence  $\frac{\sigma_{n_k}}{n_k}$  converge to  $\frac{1}{4\ell} = \frac{\sqrt{3}}{2}$ . Moreover  $\frac{\alpha_n}{\sqrt{n}}$  is bounded, then we can subtracted from  $\frac{\alpha_n}{\sqrt{n}}$  a sequences which converges to some  $y$ , using the convergence of the sequence  $\frac{b_{n_k}}{\sqrt{n_k}}$  to  $x$ , and equations 2) and 4) of the proposition, one gets from equation 2)

$$6\ell^2 = \frac{\ell^4}{y} + \frac{\ell^4}{y} + y,$$

and from 4)

$$y = 0,$$

which give a contradiction.

**Second case.**  $x = 0$ . In such a case, we obtain from statements 3) and 4) and boundedness of  $\frac{\alpha_n}{\sqrt{n}}$ , one gets

$$\lim_{k \rightarrow \infty} \frac{\sigma_{n_k}}{n_k} = \frac{1}{4\ell} + \ell,$$

and

$$\lim_{k \rightarrow \infty} \frac{\sigma_{n_k}}{n_k} = 0,$$

This give a contradiction.

The only real solution of equation (3.10) is  $x = \frac{1}{\sqrt{3}}$ . Hence the unique accumulation point of the bounded sequence  $\frac{b_n}{\sqrt{n}}$  is  $\frac{1}{\sqrt{3}}$ .

**The sequence  $\sigma_n$ .** Since  $\sigma_n = \frac{n}{4c_n} + c_{n-2} - b_{n-2}$ , hence

$$\lim_{n \rightarrow \infty} \frac{\sigma_n}{\sqrt{n}} = \frac{1}{4\ell} + \ell - 2\ell = \frac{1}{\sqrt{3}}.$$

**The sequence  $\alpha_n$ .** Letting  $n$  to infinity in the following equation

$$c_{n+2}c_{n+1}\frac{\sigma_{n+2}}{b_{n+2}} + b_n\sigma_n + \alpha_n = \frac{n}{2} + \frac{1}{4},$$

one gets

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{n} = \frac{1}{2} - \ell^2 - 4\ell^2 = \frac{1}{12}.$$

**7) The sequence  $\delta_n$ .** From the relation

$$\delta_n = \frac{c_n c_{n-1} \sigma_n}{b_n},$$

as  $n$  goes to infinity one gets

$$\lim_{n \rightarrow \infty} \frac{\delta_n}{n} = \ell^2 = \frac{1}{12}.$$

**The sequence  $\widehat{\frac{k_n}{k_n}}$ .** Since

$$\lim_{n \rightarrow \infty} \frac{\widehat{k_n}}{k_n} = \lim_{n \rightarrow \infty} \frac{\sigma_n}{b_n} = 1.$$

Which complete the proof.

**Theorem 3.3** *The asymptotic behavior*

$$\lim_{n \rightarrow \infty} \frac{P_n(\sqrt[4]{n}x)}{Q_n(\sqrt[4]{n}x)} = 2\sqrt{3} \left( \frac{x\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)}{1 + \varphi^2\left(\sqrt[4]{\frac{3}{4}}x\right)} \right)^2,$$

hold uniformly on compact subset of  $\mathbb{C} \setminus [-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ , where

$\varphi(x) = x + \sqrt{x^2 - 1}$ , with  $\sqrt{x^2 - 1} > 0$  for  $x > 1$ , i.e., the conformal mapping of  $\mathbb{C} \setminus [-1, 1]$  onto the exterior of the closed unit disk.

**Proof.** Since from the third relation in proposition 2.2, we have

$$x^2 Q_n(x) = P_{n+2}(x) + \sigma_n P_n(x) + \delta_n P_{n-2}(x),$$

thus

$$x^2 \frac{Q_n(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{P_{n+2}(\sqrt[4]{n}x)}{\sqrt[4]{n}P_{n+1}(\sqrt[4]{n}x)} \frac{P_{n+1}(\sqrt[4]{n}x)}{\sqrt[4]{n}P_n(\sqrt[4]{n}x)} + \frac{\sigma_n}{\sqrt{n}} + \frac{\delta_n}{n} \frac{\sqrt[4]{n}P_{n-2}(\sqrt[4]{n}x)}{P_{n-1}(\sqrt[4]{n}x)} \frac{\sqrt[4]{n}P_{n-1}(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)},$$

using the fact that  $\frac{\sigma_n}{\sqrt{n}} \rightarrow \frac{1}{\sqrt{3}}$ ,  $\frac{\delta_n}{\sqrt{n}} \rightarrow \frac{1}{12}$  and

$$\lim_{n \rightarrow \infty} \frac{\sqrt[4]{n} P_{n-1}(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{\sqrt[4]{12}}{\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)},$$

uniformly on compact subset of  $\mathbb{C} \setminus [-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ , one gets

$$\lim_{n \rightarrow \infty} x^2 \frac{Q_n(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{\left(\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)\right)^2}{\sqrt{12}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{12}\left(\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)\right)^2} = \frac{\left(1 + \left(\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)\right)^2\right)^2}{\sqrt{12}\left(\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)\right)^2},$$

which complete the proof.

**Remark 3.4**

1. For all  $n$  the zeros of  $Q_{2n}$  interlaces with the zeros of  $P_{2n}$ .
2. What can say about the zeros of the polynomial  $Q_{2n+1}$  compared with those of  $P_{2n+1}$ ?

The proof of the first item is like the proof of the case  $\lambda_2 = 0$ .

Consider the non monic Sobolev orthogonal polynomials  $\widehat{Q}_n$ , with norm equal one.  $\widehat{Q}_n(x) = c_n x^n + b_{n-2} x^{n-2} + \dots$

**Proposition 3.5**

1. For all  $n \geq 0$ , the polynomial  $\widehat{Q}_n$ , satisfies the three term recurrence relation

$$x^2 \widehat{Q}_n(x) = \alpha_n \widehat{Q}_{n+2}(x) + \beta_n \widehat{Q}_n(x) + \alpha_{n-2} \widehat{Q}_{n-2}(x),$$

with  $Q_{-2}(x) = 0$ ,  $Q_{-1}(x) = 0$ .

2. The zeros of  $\widehat{Q}_n$  interlaces with the zeros of  $\widehat{Q}_{n-2}$ .

The proof of the proposition is like the proof for the classical orthogonal polynomial case.

## 4 General case

Let  $r \in \mathbb{N}$ ,  $\lambda_k \geq 0$  for  $k \in \{1, \dots, r\}$ , and defined the Sobolev inner product by

$$\langle f, g \rangle_S = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^4} dx + \sum_{k=0}^r \lambda_k f^{(k)}(0)g^{(k)}(0).$$

Let  $Q_{n,r}$  be the monic Sobolev orthogonal polynomials with respect to the inner product defined above.

**Prediction 4.1** *On every compact subset of  $\mathbb{C} \setminus [-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ , we have*

$$\lim_{n \rightarrow \infty} \frac{P_n(\sqrt[4]{n}x)}{Q_{n,r}(\sqrt[4]{n}x)} = \left( \sqrt[4]{12} \frac{x \varphi\left(\sqrt[4]{\frac{3}{4}}x\right)}{1 + \varphi^2\left(\sqrt[4]{\frac{3}{4}}x\right)} \right)^{r+1},$$

*the convergence hold uniformly.*

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Address: College of Applied Sciences Umm Al-Qura University P.O Box (715), Makkah, Saudi Arabia.

E-mail: bouali25@laposte.net & mabouali@uqu.edu.sa